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Lattices compatible with regular polytopes

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For the 80th birthday of Ludwig Danzer

Abstract

In a recent paper, Karpenkov has classified the lattice polytopes (that is, with vertices in the integer lattice $\mathbb{Z}^d$) which are regular with respect to those affinities which preserve the lattice. An alternative approach is adopted in this paper. For each regular polytope P in euclidean space $\mathbb{E}^d$, those lattices Λ are classified which are compatible with P, in the sense that some translate of Λ contains the vertices of P, and this translate is preserved by the symmetries of P.

1. Introduction

A convex d-polytope P whose vertex-set vert P is a subset of the integer lattice $\mathbb{Z}^d$ in euclidean space $\mathbb{E}^d$ is called *lattice regular* if it is regular with respect to the group $\mathcal{U}_d$ of affinities which preserve $\mathbb{Z}^d$. Recall that being *regular* means that the subgroup $\mathcal{L}(P)$ of $\mathcal{U}_d$ which preserves P is transitive on the family $\mathcal{F}(P)$ of *flags* of P, which are chains $F_0 < F_1 < \cdots < F_{d-1}$ of faces of P. Karpenkov [2] has recently classified such lattice regular polytopes, up to equivalence under $\mathcal{U}_d$.

Here, we pursue an alternative approach to this problem. We start with a convex polytope P which is regular in the usual sense, with respect to the group $\mathcal{I}_d$ of isometries of $\mathbb{E}^d$, and ask which lattices Λ are *compatible* with P, meaning that (up to translation) vert$P \subseteq \Lambda$, and the subgroup $\mathcal{G}(P)$ of $\mathcal{I}_d$ of (isometric) symmetries of P preserves Λ. The advantage of looking at things from this viewpoint is that we no longer have to concern ourselves with the non-uniqueness arising from the equivalence under $\mathcal{U}_d$ in the original problem.

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The outline of the rest of the paper is as follows. In Section 2, we consider the problem in general terms, in particular introducing the edge-module $\Lambda_e = \Lambda_e(P)$ of a polytope P. In Section 3, we eliminate the non-crystallographic cases, where Λ_e is non-discrete. In the following Sections 4–6, we treat the three infinite families of regular polytopes, namely, the simplices, cubes and cross-polytopes. In Section 7, we conclude the classification by dealing with the regular hexagon and 24-cell. Finally, we summarize the results in Section 8.

General references for the background to regular polytope theory are [1,6]; we shall usually follow the notational conventions of the latter.

2. General considerations

We begin by remarking that our rephrasing of the problem is equivalent to that of Karpenkov [2]. A finite group of affinities (such as a subgroup of $\mathcal{U}_d$) is conjugate under a suitable affinity Φ to a group of isometries, and so, if a convex polytope P is lattice regular, then some affine image ΦP of P is actually regular in the usual sense (compare [3]), and the corresponding image $\Lambda = \Phi\mathbb{Z}^d$ of $\mathbb{Z}^d$ is a lattice which is compatible with ΦP.

Conversely, if we are given a regular polytope P in $\mathbb{E}^d$ and a lattice Λ compatible with P, then we can apply an affinity Ψ taking a basis of Λ into the standard basis of $\mathbb{E}^d$ (and thus Λ into $\Psi\Lambda = \mathbb{Z}^d$), so that ΨP will be lattice regular in the sense of [2].

We thus start with a convex d-polytope P in $\mathbb{E}^d$ which is regular in the usual sense, under the group $\mathcal{G} = \mathcal{G}(P)$ of isometries which preserve P. (We shall always make the assumption that polytopes have dimension d, and lattices have rank d.) As in Section 1, we say that a lattice Λ in $\mathbb{E}^d$ is *compatible* with P if there is some translate $\Lambda' = \Lambda + t$ of Λ such that $\operatorname{vert} P \subseteq \Lambda'$ and $\mathcal{G}\Lambda' = \Lambda'$. (We phrase things this way, because it will sometimes be convenient not to have the vertices of P sitting in an actual lattice.)

The *edge-module* $\Lambda_e = \Lambda_e(P)$ is defined as follows: if

$$\mathcal{E} = \mathcal{E}(P) := \{v - w \mid v, w \in \operatorname{vert} P\},$$

then $\Lambda_e := \mathbb{Z}[\mathcal{E}]$ is the $\mathbb{Z}$-module generated by $\mathcal{E}$. In the definition of Λ_e, it is clear that we need only take as generators the *edge-vectors* $v - w$, where v, w are the two vertices of an edge of P; this accounts for the nomenclature. If Λ, Λ' are lattices such that Λ is a sublattice of Λ', then we say that Λ' is a *superlattice* of Λ. We first have an obvious result.

Theorem 2.1. *If Λ is a lattice which is compatible with the regular polytope P, then Λ is a superlattice of $\Lambda_e(P)$.*

Proof. Indeed, Λ must contain all edge-vectors in $\mathcal{E}(P)$, and the claim follows at once. □

Remark 2.2. Of course, there are extra conditions imposed on such a superlattice Λ by the symmetries of P.

Recall that a vector v in a lattice Λ is *primitive* if $v \neq kw$ for any $w \in \Lambda$ and (integer) $k \geqslant 2$. We call a lattice Λ which is compatible with a regular polytope P *primitive* if each edge-vector in $\mathcal{E}(P)$ is primitive in Λ. If $q \in \mathbb{N}$, then

$$\frac{1}{q}\Lambda := \left\{\frac{1}{q}v \mid v \in \Lambda\right\}$$

has the obvious meaning. It is clear that a lattice Λ which is compatible with P is of the form $\Lambda = \frac{1}{q}\Lambda'$ for some $q \in \mathbb{N}$ and some primitive compatible lattice Λ'; thus it suffices to determine the primitive lattices.

The subsequent discussion is aided by the following remark.

Proposition 2.3. *Let Λ be a lattice in $\mathbb{E}^d$ for some d, and let Λ' be a superlattice of Λ. Then Λ' is a sublattice of $\frac{1}{q}\Lambda$ for some q.*

Proof. A basis of Λ consists of integer linear combinations of vectors of a basis of Λ'. It follows at once that a basis of Λ' consists of rational linear combinations of vectors of a basis of Λ. The conclusion of the proposition is immediate. □

Last in this section, we recall that the *reciprocal lattice* of Λ is

$$\Lambda^* = \{y \in \mathbb{E}^d \mid \langle x, y\rangle \in \mathbb{Z} \text{ for all } x \in \Lambda\}.$$

Note that $\Lambda^* \cong \mathrm{Hom}(\Lambda, \mathbb{Z})$ (as abelian groups). We shall see that, if Λ is compatible with some regular polytope P, then a suitable multiple of Λ^* will be as well.

3. The non-crystallographic cases

The edge-module Λ_e of a regular polytope P is generated by finitely many vectors, and so is of finite rank over $\mathbb{Z}$. However, it is not necessarily a lattice, by which we mean that it is discrete. In a more general context (see [4]), we have called polytopes for which Λ_e is discrete *rational*; in the convex case, the polytopes are usually referred to as *crystallographic*. In this section, we dispose of the non-crystallographic cases.

We begin with the regular polygons $\{p\}$ for $p \geqslant 3$.

Theorem 3.1. *The edge-module $\Lambda_e(P)$ of a regular p-gon P is a lattice if and only if $p = 3,\ 4$ or 6.*

Proof. If Λ_e is a lattice in $\mathbb{E}^2$, then it has a shortest vector v, say. We lose no generality in scaling so that $\|v\| = 1$. Let Φ_p denote rotation about the origin o through the angle $2\pi/p$, so that Φ_p is a symmetry of Λ_e. For $p \geqslant 7$, we have

$$\|\Phi_p v - v\| = 2\sin\frac{\pi}{p} < 1,$$

contradicting the definition of v. For $p = 5$, we have

$$\|\Phi_5^2 v + v\| = 2\sin\frac{\pi}{10} < 1,$$

again a contradiction. Thus all cases except for $p = 3,\ 4$ or 6 have been excluded; that these are crystallographic is well known, and will in any event be dealt with later. □

The remaining non-crystallographic regular polytopes are those with 5-fold symmetries.

Theorem 3.2. *The edge-module $\Lambda_e(P)$ is non-discrete if P is a dodecahedron, icosahedron, 120-cell or 600-cell.*

Proof. Each of the dodecahedron and 120-cell has (2-dimensional) faces which are regular pentagons. The links of a vertex of an icosahedron and edge of a 600-cell are also regular pentagons. Thus, in every case, the edge-module contains a submodule congruent to the edge-module of a regular pentagon; this is non-discrete by Theorem 3.1, which implies the claim of the theorem. □

4. Simplices

The case of the regular d-simplex T is most easily treated in $\mathbb{E}^{d+1}$. We choose the vertices of T to be the standard basis vectors $e_0, \ldots, e_d$ of $\mathbb{E}^{d+1}$. The symmetry group $\mathcal{G} = \mathcal{G}(T)$ of T thus consists of all permutations of $e_0, \ldots, e_d$, and so of all permutations of the coordinates of $\mathbb{E}^{d+1}$. We see at once that we have

Theorem 4.1. *The edge-module $\Lambda_e(T)$ of the regular d-simplex T is*

$$\begin{aligned}\Lambda_1 &:= \langle e_i - e_j \mid 0 \leqslant i < j \leqslant d\rangle \\ &= \{(\xi_0, \ldots, \xi_d) \in \mathbb{Z}^{d+1} \mid \xi_0 + \cdots + \xi_d = 0\}.\end{aligned}$$

Ignoring the scaling factor $1/q$ in the previous discussion, our next task is to determine the sublattices Λ of $\Lambda_e = \Lambda_1$ which have the symmetry of T. Our arguments exactly parallel those of [5]; see also [6, Sections 6D,6E]. Let s be the smallest positive difference $\alpha_i - \alpha_j$ which occurs in vectors $a = (\alpha_0, \ldots, \alpha_d) \in \Lambda$. Permuting coordinates as necessary, we can assume that $i = 0$, $j = 1$ and that $s = \alpha_0 - \alpha_1$. If a' is obtained from a by the transposition (0 1) which interchanges the first two coordinates, then

$$a - a' = (s, -s, 0^{d-1}),$$

where r^k in a vector denotes a string $r, \ldots, r$ of length k. Again applying the symmetries of T, we see that $s\Lambda_1$ is a sublattice of Λ.

We now add to a suitable multiples of the vectors $s(e_0 - e_j)$, so as to ensure that the (new) coordinates α_j satisfy $-s < \alpha_j \leqslant 0$ for $j = 1, \ldots, d$. It is immediate that all such α_j are equal; otherwise, we have a smaller difference than s between two coordinates of a vector in Λ. If all these $\alpha_j = 0$ (with all possible choices of a), then $\Lambda = s\Lambda_1$; bear in mind that Λ is a sublattice of Λ_1. Alternatively, Λ contains a vector rb for some $r \in \mathbb{N}$, where $b = (d, (-1)^d)$ (with the convention introduced before). In fact, if we write Λ_b for the sublattice of Λ_1 generated by b and its images under the symmetries of T, then we actually have $\Lambda \geqslant r\Lambda_b$, and $s = (d+1)r$. We further note that, if $b' := (-1, d, (-1)^{d-1})$, then $b' \in \Lambda_b$, and $b - b' = (d+1)(1, -1, 0^{d-1})$; we conclude that

$$(d+1)\Lambda_1 \leqslant \Lambda_b. \tag{4.1}$$

Now let us revert to the original problem. A lattice Λ which is compatible with T contains $\frac{1}{q}\Lambda_1$ for some $q \in \mathbb{N}$ (q is not fractional, because Λ contains Λ_1), and possibly some $r\Lambda_b$ with a suitable fractional r. If Λ is primitive, then $q = 1$, and we can write

$$\Lambda = r\Lambda_b \qquad \text{or} \qquad \Lambda_1 + r\Lambda_b$$

for some (positive) rational r (recall (4.1)). If $r = p/q$ in its lowest terms (not the same q as before), then $kp \equiv 1 (\mathrm{mod}\ q)$ for some $k \in \mathbb{N}$, and since $kr\Lambda_b \leqslant r\Lambda_b$ and $\Lambda_b \leqslant \Lambda_1$, we can therefore assume that $r = 1/q$ for some $q \in \mathbb{N}$. Moreover, by (4.1), it follows that $(d+1)r\Lambda_1 = r\Lambda_b \leqslant \Lambda$. Hence, if Λ is to be primitive, then $(d+1)r$ cannot be fractional; it follows that $(d+1)r = (d+1)/q \in \mathbb{N}$, so that q is a divisor of $d+1$.

Summarizing this discussion, we see that we have proved

Theorem 4.2. *The primitive lattices which are compatible with the regular d-simplex T are*

$$\begin{cases} \Lambda_1, \\ \Lambda_q := \Lambda_1 + \dfrac{1}{q}\Lambda_b, & \textit{for some } q|(d+1) \textit{ with } 1 < q < d+1, \\ \Lambda_{d+1} := \dfrac{1}{d+1}\Lambda_b. \end{cases}$$

There is a comparatively easy way of getting at the indices of $\Lambda_e = \Lambda_1$ in these superlattices. Define

$$\Gamma_k := \{(\xi_0, \ldots, \xi_d) \in \mathbb{Z}^{d+1} \mid \xi_0 + \cdots + \xi_d = k\};$$

thus $\Lambda_1 = \Gamma_0$. Then it is clear that

$$\Gamma_k + \Gamma_m = \Gamma_{k+m}$$

for each k and m. Next, define $H_k := \operatorname{aff} \Gamma_k$ to be the hyperplane containing Γ_k. If Π denotes the orthogonal projection onto H_0, then $\Pi\Gamma_{d+1} = \Lambda_1$. Further, we see that

$$\Pi e_0 = \frac{1}{d+1} b,$$

whose images under $\mathcal{G}$ generate Λ_{d+1}. It is then an easy matter to conclude that

$$[\Lambda_q : \Lambda_1] = q$$

for each divisor q of $d+1$.

We end the section by observing that $\Lambda_{d+1} = \Lambda_1^*$. Indeed, we actually have

Proposition 4.3. *The primitive lattices compatible with the regular simplex occur in reciprocal pairs:*

$$\Lambda_q^* = \Lambda_{(d+1)/q}$$

for each $q|(d+1)$.

5. Cubes

For the d-cube C, the procedure is more straightforward than that of Section 4. The standard unit cube has vertex-set $\{0, 1\}^d$, and we see at once that

$$\Lambda_e = \Lambda_1 := \mathbb{Z}^d.$$

The group $\mathcal{G} = \mathcal{G}(C)$ of C consists (up to translations) of all permutations of the coordinates of $\mathbb{E}^d$, together with all changes of their signs. We employ similar arguments to those of Section 4 (see [5] or [6, Section 6D]), but for completeness it is worth briefly recalling the details here.

We are thus looking for sublattices of $\mathbb{Z}^d$ on which $\mathcal{G}$ acts; we may assume that $d \geqslant 2$. Let $\Lambda \leqslant \mathbb{Z}^d$; in the present case, we let $a = (\alpha_1, \ldots, \alpha_d) \in \Lambda$ be a vector with smallest positive coordinate $s = \alpha_j$. Under the action of $\mathcal{G}$, we can suppose that $j = 1$; changing the sign of the first coordinate and subtracting the new vector from the old shows that $(2s, 0^{d-1}) \in \Lambda$. Reducing each coordinate of a modulo $2s$ now shows that Λ is generated by the vector $(s^k, 0^{d-k})$ and its images under $\mathcal{G}$, for some $k \geqslant 1$. However, if $3 \leqslant k < d$, then $(0, s^k, 0^{d-k-1}) \in \Lambda$ also, which leads quickly to $(s, s, 0^{d-2}) \in \Lambda$ (and then, if k is odd, to $(s, 0^{d-1}) \in \Lambda$ as well).

In conclusion, we have shown that the sublattices of Λ_1 which are preserved by $\mathcal{G}$ are $s\Lambda_1$, $s\Lambda_2$ or $s\Lambda_d$, generated respectively by

$$s(1, 0^{d-1}), \quad s(1, 1, 0^{d-2}) \quad \text{or} \quad s(1^d) \tag{5.1}$$

and their images under $\mathcal{G}$, for some $s \in \mathbb{N}$ (of course, $\Lambda_2 = \Lambda_d$ when $d = 2$; for convenience, we have changed the notation from [5,6]). Apart from the obvious subgroup relationships, we have

$$\begin{aligned} \Lambda_2 &\leqslant \Lambda_1, &\quad& \text{with index } [\Lambda_1 : \Lambda_2] = 2, \\ \Lambda_d &\leqslant \Lambda_1, && \text{with index } [\Lambda_1 : \Lambda_d] = 2^{d-1}, \end{aligned}$$

and, if d is even,

$$\Lambda_d \leqslant \Lambda_2, \qquad \text{with index } [\Lambda_2 : \Lambda_d] = 2^{d-2}.$$

Further, we note that

$$[\Lambda_2 : 2\Lambda_1] = 2^{d-1}, \qquad [\Lambda_d : 2\Lambda_1] = 2.$$

Finally, Λ_1 is self-reciprocal, while

$$\Lambda_2^* = \frac{1}{2}\Lambda_d.$$

It is now an easy matter to deduce

Theorem 5.1. *The primitive lattices which are compatible with the regular d-cube C are*

$$\begin{cases} \Lambda_1, \\ \dfrac{1}{2}\Lambda_d, \\ \dfrac{1}{2}\Lambda_2. \end{cases}$$

6. Cross-polytopes

The analysis for the regular d-cross-polytope X is very similar to that for the cube. Taking the vertices of X to be all $\pm e_j$ for $j = 1, \dots, d$, with $\{e_1, \dots, e_d\}$ the standard basis of $\mathbb{E}^d$, we see at once that $\Lambda_e = \Lambda_e(X) = \Lambda_2$ in the notation of Section 5. We have already detailed the subgroup relationships in that section (it is helpful to bear in mind that $\Lambda_2 \leqslant \Lambda_1$, and so the previous analysis applies), and the main result follows quickly.

Theorem 6.1. *The primitive lattices which are compatible with the regular d-cross-polytope X are*

$$\begin{cases} \Lambda_2, \\ \Lambda_1, \\ \dfrac{1}{2}\Lambda_d, \end{cases}$$

with Λ_r as defined by (5.1).

7. The anomalous cases

There remain two polytopes to be considered: these are the hexagon and the 24-cell.

We can treat the hexagon H most easily by observing that it is the Minkowski (vector) sum $T - T$ of a regular triangle T and its opposite $-T$. This tells us at once that its edge-module $\Lambda_e(H)$ is the same as that of T, namely, Λ_1 in the notation of Section 4 (with $d = 2$). The only other divisor of $2 + 1 = 3$ is 3, and we quickly conclude that

Theorem 7.1. *The primitive lattices which are compatible with the regular hexagon H are the reciprocal lattices*

$$\begin{cases} \Lambda_1, \\ \dfrac{1}{3}\Lambda_b, \end{cases}$$

in the notation of Section 4.

Remark 7.2. Note that the centre of the hexagon H is a point of $\Lambda_e(H)$.

We finally come to the 24-cell V. For simplicity of notation, we take its vertices to be

$$\pm 2e_j \quad \text{for } j = 1, \ldots, 4, \qquad (\pm 1, \pm 1, \pm 1, \pm 1),$$

with $\{e_1, \ldots, e_4\}$ the standard basis of $\mathbb{E}^4$. Then its edge-module is Λ_4 in the notation of Section 5. Now Λ_1 does not have the symmetry of V, whereas Λ_2 does; the latter is the edge-module of the 24-cell with vertices

$$\pm e_j \pm e_k \quad \text{for } 1 \leqslant j < k \leqslant 4.$$

Note that Λ_4 is similar to Λ_2, but $\sqrt{2}$ times as large. Comparing [5] or [6, Section 6E] shows that (up to scaling) these two lattices are the only ones with the symmetry of V, and thence it is straightforward to deduce

Theorem 7.3. *The primitive lattices which are compatible with the regular 24-cell V are*

$$\begin{cases} \Lambda_4, \\ \Lambda_2, \end{cases}$$

with Λ_r as defined by (5.1) with $d = 4$.

Remark 7.4. Note that the 24-cell V has diametral hexagons, and hence the centre of V is a point of $\Lambda_e(V)$. We have already observed that Λ_4 and $\frac{1}{2}\Lambda_2$ are reciprocal lattices.

8. Summary of results

We end with a table which summarizes the results obtained in the paper. Here, $\ell(P)$ is the number of primitive lattices which are compatible with the regular polytope P, and $\delta(k)$ denotes the number of divisors of $k \in \mathbb{N}$.

d	polytope P	$\ell(P)$
1	segment	1
2	triangle	2
	square	2
	hexagon	2
$d \geqslant 3$	simplex	$\delta(d+1)$
	cube	3
	cross-polytope	3
4	24-cell	2

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